

ON REFLECTIVE-COREFLECTIVE EQUIVALENCE AND ASSOCIATED PAIRS

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ABSTRACT. We show that a reflective/coreflective pair of full subcategories satisfies a “maximal-normal”-type equivalence if and only if it is an associated pair in the sense of Kelly and Lawvere.

1. INTRODUCTION

In a recent paper [1] we explored a special type of category equivalence between reflective/coreflective pairs of subcategories that we first encountered in the context of crossed-product duality for C^* -algebras. Because our main example of this phenomenon involved categories of maximal and normal C^* -coactions of locally compact groups, we called it a “maximal-normal”-type equivalence.

Since then, F. W. Lawvere has drawn our attention to [3], where G. M. Kelly and he introduced the concept of *associated pairs* of subcategories. The purpose of this short note is to show that these two notions of equivalence are the same: a reflective/coreflective pair of full subcategories satisfies the “maximal-normal”-type equivalence considered in [1] if and only if it is an associated pair in the sense of [3].

As operator algebraists, we had hoped with [1] to initiate a cross-fertilization between operator algebras and category theory, and we are grateful to Ross Street for the role he has played in helping this happen. Our understanding of the operator-algebraic examples has certainly been deepened by this connection; ideally, the techniques and examples of “maximal-normal”-type equivalence will in turn provide a way of looking at associated pairs that will also be useful to category theorists.

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2. MAXIMAL-NORMAL EQUIVALENCES AND ASSOCIATED PAIRS

Our conventions regarding category theory follow [4]; see also [1]. Throughout this note, we let \mathcal{M} and \mathcal{N} denote full subcategories of a category \mathcal{C} , with \mathcal{N} reflective and \mathcal{M} coreflective. The inclusion functors $I : \mathcal{M} \rightarrow \mathcal{C}$ and $J : \mathcal{N} \rightarrow \mathcal{C}$ are then both full and faithful. We also use the following notation:

- $N : \mathcal{C} \rightarrow \mathcal{N}$ is a reflector and $\theta : 1_{\mathcal{C}} \rightarrow JN$ denotes the unit of the adjunction $N \dashv J$;
- $M : \mathcal{C} \rightarrow \mathcal{M}$ is a coreflector and $\psi : IM \rightarrow 1_{\mathcal{C}}$ denotes the counit of the adjunction $I \dashv M$.

In [1, Corollary 4.4] we showed that the adjunction $NI \dashv MJ$ is an adjoint equivalence between \mathcal{M} and \mathcal{N} if and only if

- (I) for each $y \in \text{Obj } \mathcal{N}$, (y, ψ_y) is an initial object in the comma category $My \downarrow \mathcal{N}$; and
- (F) for each $x \in \text{Obj } \mathcal{M}$, (x, θ_x) is a final object in the comma category $\mathcal{M} \downarrow Nx$.

In all our examples in [1], the adjoint equivalence $NI \dashv MJ$ between \mathcal{M} and \mathcal{N} was what we called the “maximal-normal” type (recall that this terminology was motivated by the particular example of maximal and normal coactions on C^* -algebras; see [1, Corollary 6.16]): in addition to (I) and (F), such an adjunction satisfies

- (A) for each $z \in \text{Obj } \mathcal{C}$, $(Nz, \theta_z \circ \psi_z)$ is an initial object in $Mz \downarrow \mathcal{N}$.

Equivalently, by [1, Theorem 3.4], (I) and (F) hold, and

- (B) for each $z \in \text{Obj } \mathcal{C}$, $(Mz, \theta_z \circ \psi_z)$ is a final object in $\mathcal{M} \downarrow Nz$.

In fact, conditions (A) and (B) alone suffice:

Proposition 2.1. *The adjunction $NI \dashv MJ$ between \mathcal{M} and \mathcal{N} is a “maximal-normal” adjoint equivalence if and only if (A) and (B) hold.*

Proof. By [1, Theorem 4.3], (I) is equivalent to

- (I') for each $y \in \text{Obj } \mathcal{N}$, $N\psi_y : NMy \rightarrow Ny$ is an isomorphism,

while (F) is equivalent to

- (F') for each $x \in \text{Obj } \mathcal{M}$, $M\theta_x : Mx \rightarrow MNx$ is an isomorphism.

On the other hand, by [1, Theorem 3.4], (A) is equivalent to

- (A') for each $z \in \text{Obj } \mathcal{C}$, $N\psi_z$ is an isomorphism,

while (B) is equivalent to

- (B') for each $z \in \text{Obj } \mathcal{C}$, $M\theta_z$ is an isomorphism.

Now clearly, (A') implies (I') and (B') implies (F'), so (A) implies (I) and (B) implies (F). \square

We now recall from [2, 3] that a morphism f in $\mathcal{C}(x, y)$ and an object z of \mathcal{C} are said to be *orthogonal* when the map $\Phi_{f,z}$ from $\mathcal{C}(y, z)$ into $\mathcal{C}(x, z)$ given by $\Phi_{f,z}(g) = g \circ f$ is a bijection. The collection of all morphisms in \mathcal{C} that are orthogonal to every object of \mathcal{N} is denoted by \mathcal{N}^\perp .

As shown in [3, Proposition 2.1], a morphism $f : x \rightarrow y$ in \mathcal{C} belongs to \mathcal{N}^\perp if and only if f is inverted by N , that is, Nf is an isomorphism. (The standing assumption in [3] that \mathcal{N} is replete is not necessary for this fact to be true. To see this, note that Nf is an isomorphism if and only if the map $\Psi_{f,z}$ from $\mathcal{N}(Ny, z)$ into $\mathcal{N}(Nx, z)$ given by $\Psi_{f,z}(h) = h \circ Nf$ is a bijection for each object z of \mathcal{N} . For each such z , the universal properties of θ imply that the map $\tau_{w,z}$ from $\mathcal{N}(Nw, z)$ into $\mathcal{C}(w, z)$ given by $\tau_{w,z}(g) = g \circ \theta_w$ is a bijection for each object w of \mathcal{C} . Now, as $\theta_y \circ f = Nf \circ \theta_x$, the diagram

$$\begin{array}{ccc} \mathcal{N}(Ny, z) & \xrightarrow{\Psi_{f,z}} & \mathcal{N}(Nx, z) \\ \tau_{y,z} \downarrow & & \downarrow \tau_{x,z} \\ \mathcal{C}(y, z) & \xrightarrow{\Phi_{f,z}} & \mathcal{C}(x, z) \end{array}$$

is readily seen to commute. It follows that $\Psi_{f,z}$ is a bijection if and only if $\Phi_{f,z}$ is a bijection. This shows that Nf is an isomorphism if and only if f is orthogonal to z for each object z of \mathcal{N} , *i.e.*, if and only if f belongs to \mathcal{N}^\perp .)

Similarly, a morphism f in $\mathcal{C}(x, y)$ and an object z in \mathcal{C} are *co-orthogonal* when the map $g \rightarrow f \circ g$ from $\mathcal{C}(z, x)$ into $\mathcal{C}(z, y)$ is a bijection. The collection of all morphisms in \mathcal{C} that are co-orthogonal to every object in \mathcal{M} is denoted by \mathcal{M}^\top . Equivalently, a morphism $f : x \rightarrow y$ in \mathcal{C} belongs to \mathcal{M}^\top if and only if f is inverted by M , that is, if and only if Mf is an isomorphism.

The pair $(\mathcal{N}, \mathcal{M})$ is called an *associated pair* if $\mathcal{N}^\perp = \mathcal{M}^\top$; equivalently, if for every morphism f in \mathcal{C} , N inverts f if and only if M does. We refer to [3, Section 2] for more information concerning this concept (in the case where both \mathcal{M} and \mathcal{N} are also assumed to be replete).

Theorem 2.2. *The adjunction $NI \dashv MJ$ is a “maximal-normal” adjoint equivalence if and only if $(\mathcal{N}, \mathcal{M})$ is an associated pair.*

Proof. First assume that $(\mathcal{N}, \mathcal{M})$ is an associated pair, and let x be an object in \mathcal{C} . As pointed out above, the map $\tau_{x,z}$ is a bijection from $\mathcal{N}(Nx, z)$ into $\mathcal{C}(x, z)$ for each object z of \mathcal{N} . But $\Phi_{\theta_x, z} = \tau_{x,z}$, so this means that θ_x lies in \mathcal{N}^\perp , and therefore in \mathcal{M}^\top . As \mathcal{M}^\top consists of the morphisms in \mathcal{C} that are inverted by M , we deduce that $M\theta_x$ is

an isomorphism. This shows that (B') holds, and therefore that (B) holds. The argument that (A) holds is similar, so $NI \dashv MJ$ is a “maximal-normal” adjoint equivalence by Proposition 2.1.

Now assume that the adjunction $NI \dashv MJ$ is a “maximal-normal” adjoint equivalence. Then $N \cong NIM$ by [1, Proposition 5.3], and NI is an equivalence. So for any morphism f of \mathcal{C} , we have

$$\begin{aligned} Nf \text{ is an isomorphism} &\Leftrightarrow NIMf \text{ is an isomorphism} \\ &\Leftrightarrow Mf \text{ is an isomorphism.} \end{aligned}$$

Thus $(\mathcal{N}, \mathcal{M})$ is an associated pair. □

Remark 2.3. In the examples presented in [1, Section 6], the adjunctions $NI \dashv MJ$ are “maximal-normal” adjoint equivalences, so all the pairs $(\mathcal{N}, \mathcal{M})$ there are associated pairs. Moreover, all these pairs consist of subcategories that are easily seen to be replete. It follows from [3, Theorem 2.4] that \mathcal{M} and \mathcal{N} are uniquely determined as subcategories by each other, a fact that is not *a priori* obvious in any of the examples.

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